2.4 More properties of holomorphic functions

Definition 2.27. Let T be a topological space and A a subset. We say that $p \in A$ is an *isolated point* of A in T iff there exists a neighborhood $U \subseteq T$ of p such that $U \cap A = \{p\}$. We say that A is *discrete* in T iff all its points are isolated.

Theorem 2.28 (Riemann Continuation Theorem). *Let $D \subseteq \mathbb{C}$ be a region and $A \subset D$ a discrete and relatively closed subset. Suppose that $f \in \mathcal{O}(D \setminus A)$. Then, the following assertions are equivalent.*

1. *f extends to a holomorphic function on D.*
2. *f extends to a continuous function on D.*
3. *f is bounded in every neighborhood of any point of A.*
4. *$\lim_{z \to z_0}(z - z_0)f(z) = 0$ for each point $z_0 \in A$.*

Proof. The implications 1.⇒2.⇒3.⇒4. are clear. It remains to show 4.⇒1. It is sufficient to consider a single point $z_0 \in A$. Moreover, without loss of generality we may assume $z_0 = 0$. Since 0 is isolated, there exists an open neighborhood $U \subseteq D$ of 0 such that $U \cap A = \{0\}$. Define $g : U \to \mathbb{C}$ as follows,

$$g(z) := \begin{cases} zf(z) & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}.$$

By assumption, g is continuous in U. Define $h : U \to \mathbb{C}$ by $h(z) := zg(z)$. Since g is holomorphic in $U \setminus \{0\}$ so is h. Moreover, $h(z) = h(0) + zg(z) = h(0) + \mathrm{o}(|z|)$, so h is complex differentiable at 0 with differential $h'(0) = 0$. Thus, h is actually holomorphic in U. By Theorem 2.22 it can be represented for some radius of convergence $r > 0$ as a power series $h(z) = \sum_{n=0}^{\infty} c_n z^n$ around 0. But since $h(0) = 0$ and $h'(0) = 0$ we actually have $c_0 = 0$ and $c_1 = 0$ and thus $h(z) = z^2 \sum_{n=0}^{\infty} c_{n-2} z^n$, where the series still converges pointwise in $B_r(0)$. But since $h(z) = z^2 f(z)$ in $U \setminus \{0\}$, this implies that the power series $\sum_{n=0}^{\infty} c_{n-2} z^n$ coincides with f in $U \cap B_r(0) \setminus \{0\}$. Thus, it represents an analytic (and therefore holomorphic) extension of f to $D \cup \{0\}$. □

Theorem 2.29 (Identity Theorem). *Let D be a region and $f, g \in \mathcal{O}(D)$. The following statements are equivalent:*

1. *$f = g$*

2. *The coincidence set* $\{z \in D | f(z) = g(z)\}$ *has an accumulation point in* D.

3. *There exists a point* $z_0 \in D$ *such that* $f^{(n)}(z_0) = g^{(n)}(z_0)$ *for all* $n \in \mathbb{N}$.

Proof. The implication 1.⇒2. is trivial. We show 2.⇒3. Let $h := f - g$. Suppose $z_0 \in D$ is limit point of the coincidence set $\{z \in D | h(z) = 0\}$. Suppose there exists $m \in \mathbb{N}_0$ such that $h^{(m)}(z_0) \neq 0$ and choose the smallest such m. Since h is holomorphic in D it is also analytic by Theorem 2.22 and has a power series expansion around z_0 for some radius $r > 0$, given by

$$h(z) = \sum_{n=m}^{\infty} \frac{h^{(n)}(z_0)}{n!}(z - z_0)^n = (z - z_0)^m g(z),$$

where $g : B_r(0) \to \mathbb{C}$ is the analytic function given by the power series,

$$g(z) = \sum_{n=0}^{\infty} \frac{h^{(n+m)}(z_0)}{(n+m)!}(z - z_0)^n.$$

In particular, $g(z_0) = h^{(m)}(z_0)/m! \neq 0$. But continuity of g at z_0 implies that there must be a neighborhood $U \subseteq D$ of z_0 such that $g(z) \neq 0$ for $z \in U$. But this implies $h(z) \neq 0$ for $z \in U \setminus \{z_0\}$, a contradiction to the assumption that z_0 is a limit point of the coincidence set.

We proceed to show the implication 3.⇒1. Set $S_n := \{z \in D | h^{(n)}(z) = 0\}$ for all $n \in \mathbb{N}_0$. Then, each S_n is closed in D and so is the intersection $S := \bigcap_{n=0}^{\infty} S_n$. On the other hand, S is open since given $z_1 \in S$ the power series expansion of h around z_1 has non-zero radius r of convergence by Theorem 2.22, but is identical to zero. So every point $z \in B_r(z_1)$ is element of S. Thus S is both open and closed in D. Connectedness of D implies that S is either empty or $S = D$. The first possibility is excluded by the assumption that $z_0 \in S$. So the power series of h is zero around any point of D, hence $h = 0$, implying $f = g$ in D. □

Corollary 2.30. *Let* $I \subseteq \mathbb{R}$ *be an interval and* $f : I \to \mathbb{C}$ *some function. For any region* $D \subseteq \mathbb{C}$ *such that* $I \subset D$ *there is at most one holomorphic function* $g : D \to \mathbb{C}$ *such that* $f(z) = g(z)$ *for all* $z \in I$.

This is relevant when we are interested in extending functions on $\mathbb{R}$ or some interval $I \subset \mathbb{R}$ to holomorphic functions on the complex plane.

Proposition 2.31 (Cauchy's Estimates)**.** *Let* $D \subseteq \mathbb{C}$ *be a region,* $f \in \mathcal{O}(D)$, $z \in D$ *and* $r > 0$ *such that* $\overline{B_r(z)} \subset D$. *Then,*

$$|f^{(n)}(z)| \leq \frac{n! \|f\|_{\partial B_r(z)}}{r^n}$$

Proof. This follows by applying Proposition 2.7 to the formula of Theorem 2.22. □

Lemma 2.32 (Gutzmer Formula)**.** *Let $z_0 \in \mathbb{C}$ and $r > 0$, and suppose the power series*

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

has radius of convergence greater than r. Set $M(r) := \sup_{z \in \partial B_r(z_0)} |f(z)|$. Then,

$$\sum_{n=0}^{\infty} |c_n|^2 r^{2n} = \frac{1}{2\pi} \int_0^{2\pi} \left| f(z_0 + re^{\mathrm{i}\theta}) \right|^2 \mathrm{d}\theta \leq M(r)^2.$$

Proof. By Theorem 2.22 we have

$$c_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi\mathrm{i}} \int_0^{2\pi} \frac{f(z_0 + re^{\mathrm{i}\theta})}{(re^{\mathrm{i}\theta})^{n+1}} \mathrm{i}re^{\mathrm{i}\theta} \, \mathrm{d}\theta.$$

Equivalently,

$$c_n r^n = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{\mathrm{i}\theta}) e^{-\mathrm{i}n\theta} \, \mathrm{d}\theta.$$

Since

$$\overline{f(z_0 + re^{\mathrm{i}\theta})} = \sum_{n=0}^{\infty} \overline{c_n} r^n e^{-\mathrm{i}n\theta}$$

we have,

$$\left| f(z_0 + re^{\mathrm{i}\theta}) \right|^2 = \sum_{n=0}^{\infty} \overline{c_n} r^n f(z_0 + re^{\mathrm{i}\theta}) e^{-\mathrm{i}n\theta},$$

where the series converges uniformly as a series of functions on $\theta \in [0, 2\pi]$. Thus, we can interchange integration and summation yielding,

$$\int_0^{2\pi} \left| f(z_0 + re^{\mathrm{i}\theta}) \right|^2 \mathrm{d}\theta = \sum_{n=0}^{\infty} \overline{c_n} r^n \int_0^{2\pi} f(z_0 + re^{\mathrm{i}\theta}) e^{-\mathrm{i}n\theta} \, \mathrm{d}\theta = 2\pi \sum_{n=0}^{\infty} |c_n|^2 r^{2n}.$$

This shows the claimed equality. The stated inequality is obtained by the standard estimate of the integral. □

Theorem 2.33 (Maximum Modulus Principle)**.** *Let $D \subseteq \mathbb{C}$ be a region and $f \in \mathcal{O}(D)$. Suppose that $|f|$ has a local maximum at some point $z \in D$, i.e., that $|f(z)| = \|f\|_U := \sup_{\zeta \in U} |f(\zeta)|$ for some neighborhood $U \subseteq D$ of z, then f is constant.*

Proof. Given a point $z \in D$ and a neighborhood U of z as described, consider the power series expansion $f(\zeta) = \sum_{n=0}^{\infty} c_n(\zeta - z)^n$ of f around z. Let $\rho > 0$ such that $\overline{B_\rho(z)} \subseteq U$. Then, the power series converges with radius at least ρ and for $0 < r < \rho$ we have, by Lemma 2.32,

$$\sum_{n=0}^{\infty} |c_n|^2 r^{2n} \leq M(r)^2 \leq \|f\|_U^2 = |f(z)|^2 = |c_0|^2.$$

This implies $c_k = 0$ for all $k \in \mathbb{N}$, i.e., f is constant in $B_\rho(z)$. But then the Identity Theorem (Theorem 2.29) ensures that f is constant in all of D. $\square$

Proposition 2.34. *Let $D \subseteq \mathbb{C}$ be a bounded region and K its closure. Suppose $f : K \to \mathbb{C}$ is continuous and its restriction to D is holomorphic. Then,*

$$|f(z)| \leq \|f\|_{\partial D} \quad \forall z \in D.$$

In case of equality for some $z \in D$, f is constant.

Proof. If f is constant the inequality is an equality and is valid trivially. Thus, suppose that f is not constant. Since K is compact and f is continuous on K there exists a point $z \in K$ such that $|f(z)| = \|f\|_K$. We have to show that necessarily $z \in \partial D = K \setminus D$. Assume to the contrary that $z \in D$. Since $|f(z)| = \|f\|_K = \|f\|_D$ we can apply Theorem 2.33 with $U = D$, concluding that f is constant, a contradiction. $\square$

Theorem 2.35 (Liouville Theorem)**.** *Every bounded entire function is constant.*

Proof. Let $f \in \mathcal{O}(\mathbb{C})$ be bounded by M, i.e., $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Since f is analytic in $\mathbb{C}$ and its power series $f(z) = \sum_{n=0}^{\infty} c_n z^n$ around 0 has infinite radius of convergence. Thus, for a radius $r > 0$ we have from Lemma 2.32 the estimate,

$$\sum_{n=0}^{\infty} |c_n|^2 r^{2n} \leq M(r)^2 \leq M^2.$$

Since r can be arbitrarily large, this implies $c_k = 0$ for all $k \in \mathbb{N}$. $\square$

<u>Exercise</u> 19. Let $D \subseteq \mathbb{C}$ a region, $a \in D$. Suppose that $f \in \mathcal{O}(D \setminus \{a\})$. Show that f has a holomorphic extension to D if f' has.

<u>Exercise</u> 20. Suppose f is an entire function which satisfies

$$|f(z)| \leq a + b|z|^c \, \forall z \in \mathbb{C},$$

where a, b, c are positive constants. Show that f is a polynomial of degree less than or equal to c.

<u>Exercise</u> 21. Let f be an entire function. Show that the Taylor series of f at 0 converges uniformly in all of $\mathbb{C}$ if an only if f is a polynomial.

<u>Exercise</u> 22. Let f, g be entire functions satisfying $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. Show that there is $a \in \mathbb{C}$ such that $f = ag$.

<u>Exercise</u> 23. Let $D \subseteq \mathbb{C}$ be a region and $L \subset \mathbb{C}$ be a straight line. Let $f : D \to \mathbb{C}$ be continuous and f holomorphic in $D \setminus L$. Show that f is actually holomorphic in all of D.

<u>Exercise</u> 24. Let $D \subseteq \mathbb{C}$ be a region and $f \in \mathcal{O}(D)$. Suppose that there exists $z \in D$ such that $f^{(n)}(z) = 0$ for almost all $n \in \mathbb{N}$. Show that f is a polynomial.

<u>Exercise</u> 25. Let $D \subseteq \mathbb{C}$ be a region such that if $z \in D$ then $\overline{z} \in D$. Show that for $f \in \mathcal{O}(D)$ the following statements are equivalent:

1. $f(D \cap \mathbb{R}) \subseteq \mathbb{R}$.

2. $f(\overline{z}) = \overline{f(z)}$ for all $z \in D$.

<u>Exercise</u> 26. For each of the following properties give an example for a holomorphic function defined in some disk around 0 with that property or show that there can be no such function.

1. $f(1/n) = (-1)^n/n$ for almost all $n \in \mathbb{N}$.

2. $f(1/n) = 1/(n^2 - 1)$ for almost all $n \in \mathbb{N} \setminus \{1\}$.

3. $|f^{(n)}(0)| \geq (n!)^2$ for almost all $n \in \mathbb{N}_0$.

4. $|f(1/n)| \leq e^{-n}$ for almost all $n \in \mathbb{N}$ and $f \neq 0$.

2.5 The Open Mapping Theorem

Definition 2.36. Let X, Y be topological spaces. A map $f : X \to Y$ is called *open* iff for every open set $U \subseteq X$ the image $f(U)$ is open in Y.

Proposition 2.37 (Minimum Principle)**.** *Let $D \subseteq \mathbb{C}$ be a region and $f \in \mathcal{O}(D)$. Suppose that $|f|$ has a local minimum at some point $z \in D$, i.e., that $|f(z)| = \inf_{\zeta\in U} |f(\zeta)|$ for some neighborhood $U \subseteq D$ of z. Then, $f(z) = 0$ or f is constant in D.*

Proof. Let $z \in D$ be a local minimum and U a neighborhood of z as described. Without loss of generality we may assume that U is connected, i.e. a region. If $f(z) = 0$ we are done. Thus, suppose $f(z) \neq 0$. Since z is local minimum of $|f|$ in U, $f(\zeta) \neq 0$ for all $\zeta \in U$. So, $1/f \in \mathcal{O}(U)$. But $|1/f|$ has a local maximum at z and we may apply Theorem 2.33 to conclude that $1/f$ is constant in U. But then f is constant in U and by Theorem 2.29 constant in D. □

Proposition 2.38. *Let $D \subseteq \mathbb{C}$ be a bounded region and K its closure. Suppose $f : K \to \mathbb{C}$ is continuous and its restriction to D is holomorphic. Then, either f has zeros in D or*

$$|f(z)| \geq \inf_{\zeta\in\partial D} |f(\zeta)| \quad \forall z \in D.$$

Proof. **Exercise.** □

Lemma 2.39. *Let $D \subseteq \mathbb{C}$ be a region and $f \in \mathcal{O}(D)$. Let $z \in D$ and $r > 0$ such that $\overline{B_r(z)} \subset D$ and $2\delta := \inf_{\zeta\in\partial B_r(z)} |f(\zeta) - f(z)| > 0$. Then, $B_\delta(f(z)) \subseteq f(B_r(z))$.*

Proof. Let $a \in B_\delta(f(z))$. Then,

$$|f(\zeta) - a| \geq |f(\zeta) - f(z)| - |a - f(z)| > \delta \quad \forall \zeta \in \partial B_r(z).$$

In particular, $\inf_{\zeta\in\partial B_r(z)} |f(\zeta) - a| > |f(z) - a|$. Thus, by Proposition 2.38 $f - a$ must have zeros in the region $B_r(z)$. That is, there exists $\xi \in B_r(z)$ such that $f(\xi) = a$. □

Theorem 2.40 (Open Mapping Theorem)**.** *Let $D \subseteq \mathbb{C}$ be a region and $f \in \mathcal{O}(D)$ such that f is not constant. Then f is an open map $D \to \mathbb{C}$.*

Proof. Let $U \subseteq D$ be open. Let $z \in U$. It is enough to show that $f(U)$ contains a disc centered around $f(z)$. Since f is not constant, by the Identity Theorem (Theorem 2.29) there is a radius $r > 0$ such that $f(z) \notin f(\partial B_r(z))$ while $B_r(z) \subseteq U$. Then $2\delta := \inf_{\zeta\in\partial B_r(z)} |f(\zeta) - f(z)| > 0$ and Lemma 2.39 can be applied, showing that $B_\delta(f(z)) \subseteq f(B_r(z)) \subseteq f(U)$. □

2.6 General Cauchy Theory

We have already seen that the index $\mathrm{Ind}_\gamma(z)$ of a point z with respect to a path γ is zero, if z lies in the connected component of $\mathbb{C} \setminus |\gamma|$ which is unbounded. This motivates the following definition.

Definition 2.41. Let γ be a closed path in $\mathbb{C}$. We define the *interior* of γ as the subset $\mathrm{Int}_\gamma := \{z \in \mathbb{C} \setminus |\gamma| : \mathrm{Ind}_\gamma(z) \neq 0\}$. Similarly, we define the *exterior* of γ as the subset $\mathrm{Ext}_\gamma := \{z \in \mathbb{C} \setminus |\gamma| : \mathrm{Ind}_\gamma(z) = 0\}$.

Obviously, we have the disjoint union $\mathbb{C} = \mathrm{Int}_\gamma \cup |\gamma| \cup \mathrm{Ext}_\gamma$.

Lemma 2.42. *Let $D \subseteq \mathbb{C}$ be a region and γ a closed path in D. Suppose $g : |\gamma| \times D \to \mathbb{C}$ is a continuous function such that $z \mapsto g(\zeta, z)$ is holomorphic for all $\zeta \in |\gamma|$. Then, the function $h : D \to \mathbb{C}$ given by*

$$h(z) := \int_\gamma g(\zeta, z)\,\mathrm{d}\zeta$$

is holomorphic.

Proof. Fix $z \in D$. Let $U \subseteq D$ be a star-shaped neighborhood of z with center z (e.g. a disc centered at z). Then, for all $\zeta \in |\gamma|$ and all paths $\tilde{\gamma}$ in U we have,

$$\int_{\tilde{\gamma}} g(\zeta, z)\,\mathrm{d}z = 0,$$

by Proposition 2.11 since $z \mapsto g(\zeta, z)$ is holomorphic and thus integrable in U by Corollary 2.15. But we can interchange the order of integration by Fubini's Theorem to get

$$\int_{\tilde{\gamma}} h(z)\,\mathrm{d}z = \int_{\tilde{\gamma}} \left(\int_\gamma g(\zeta, z)\,\mathrm{d}\zeta \right) \mathrm{d}z = \int_\gamma \left(\int_{\tilde{\gamma}} g(\zeta, z)\,\mathrm{d}z \right) \mathrm{d}\zeta = 0.$$

Thus, h is integrable in U by Proposition 2.11 and therefore holomorphic in U by Corollary 2.23. Since z was arbitrary, h is holomorphic in D. □

Theorem 2.43. *Let $D \subseteq \mathbb{C}$ be a region and γ a closed path in D. Then, the following conditions are equivalent:*

1. *All $f \in \mathcal{O}(D)$ satisfy*

$$\int_\gamma f = 0.$$

2. For every $f \in \mathcal{O}(D)$ *and every* $z \in D \setminus |\gamma|$ *we have,*

$$f(z)\mathrm{Ind}_\gamma(z) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, \mathrm{d}\zeta.$$

3. $\mathrm{Int}_\gamma \subset D$.

Proof. To show 2.⇒1. for a given $f \in \mathcal{O}(D)$, choose $z \in D \setminus |\gamma|$ arbitrarily and define $h \in \mathcal{O}(D)$ via $h(\zeta) := (\zeta - z)f(\zeta)$. Applying the formula of 2. to h yields 1. since $h(z) = 0$ by construction.

We proceed to show 1.⇒3. If $D = \mathbb{C}$ there is nothing to demonstrate. So assume the contrary and let $z_0 \in \mathbb{C} \setminus D$. We have to demonstrate that $\mathrm{Ind}_\gamma(z_0) = 0$. Define $f \in \mathcal{O}(D)$ via $f(z) := (z - z_0)^{-1}$. By 1.,

$$0 = \int_\gamma f = 2\pi \mathrm{i}\, \mathrm{Ind}_\gamma(z_0).$$

It remains to demonstrate 3.⇒2. Define the function $g : D \times D \to \mathbb{C}$ as follows,

$$g(\zeta, z) := \begin{cases} \frac{f(\zeta) - f(z)}{\zeta - z} & \text{if}\, \zeta \neq z \\ f'(z) & \text{if}\, \zeta = z \end{cases}.$$

We proceed to show that g is continuous. For $(\zeta, z) \in D \times D$ such that $\zeta \neq z$ this is immediate. Thus, consider the case $\zeta = z$ and fix $z \in D$. Let $r > 0$ such that $B_r(z) \subset D$. Consider the power series expansion of f around z,

$$f(\zeta) = \sum_{n=0}^{\infty} c_n (\zeta - z)^n,$$

for all $\zeta \in B_r(z)$. Then, for $\zeta, \xi \in B_r(z)$,

$$g(\zeta, \xi) = f'(z) + \sum_{n=2}^{\infty} c_n \sum_{k=1}^{n} (\zeta - z)^{n-k} (\xi - z)^{k-1}.$$

For $\zeta, \xi \in B_\rho(z)$ with $0 < \rho < r$ we have thus the estimate,

$$|g(\zeta, \xi) - g(z, z)| \leq \left| \sum_{n=2}^{\infty} c_n \sum_{k=1}^{n} (\zeta - z)^{n-k} (\xi - z)^{k-1} \right| \leq \sum_{n=2}^{\infty} n |c_n| \rho^{n-1}.$$

However, the series on the right hand side converges for all $\rho < r$ to a continuous function which goes to 0 when $\rho \to 0$. Thus, g is continuous at (z, z). Since z was arbitrary in D this completes the proof that g is continuous in $D \times D$. Now we apply Lemma 2.42 to conclude that the function $h : D \to \mathbb{C}$ defined by

$$h(z) := \int_\gamma g(\zeta, z)\,\mathrm{d}\zeta$$

is holomorphic in D.

Now observe that for $z \in D$ we have

$$h(z) = \int_\gamma g(\zeta, z)\,\mathrm{d}\zeta = -2\pi\mathrm{i}\, f(z)\mathrm{Ind}_\gamma(z) + \int_\gamma \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta.$$

Thus, to show 2. we need to show that $h = 0$. But if $z \in D \cap \mathrm{Ext}\gamma$, then $\mathrm{Ind}_\gamma(z) = 0$ and we get

$$h(z) = \int_\gamma \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta.$$

However, this formula actually defines a holomorphic function in all of $\mathrm{Ext}\gamma$ by Lemma 2.21. Thus, we use it to extend h to a holomorphic function on the open set $D \cup \mathrm{Ext}\gamma$. Now recall that the assumption is that $\mathrm{Int}\gamma \subset D$. But this means $D \cup \mathrm{Ext}\gamma = \mathbb{C}$, i.e. h is entire. **Exercise.**Complete the proof. [Hint: Use Liouville's Theorem]. □